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Vibration damping in beams via piezo actuation using optimal boundary control

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Abstract

Open-loop optimal control theory is formulated and applied to damp out the vibrations of a beam where the control action is implemented using piezoceramic actuators. The optimal control law is derived by using a maximum principle developed for one-dimensional structures where the control function appears in the boundary conditions in the form of a moment. The objective function is specified as a weighted quadratic functional of the displacement and velocity which is to be minimized at a specified terminal time using continuous piezoelectric actuators. The expenditure of control force is included in the objective functional as a penalty term. The explicit solution of the problem is developed for cantilever beams using eigenfunction expansions of the state and adjoint variables. The effectiveness of the proposed control mechanism is assessed by plotting the displacement and velocity against time. It is shown that both quantities are damped out substantially as compared to an uncontrolled beam and this reduction depends on the magnitude of the control moment. The capabilities of piezo actuation are also investigated by means of control moment versus piezo and beam thickness graphs which indicate the required minimum level of voltage to be applied on piezo materials in relation to geometric dimensions of the combined active/passive structure. The graphs show the magnitude of the control moment which can be achieved using piezoceramics in terms of problem inputs such as voltage, piezo and beam thicknesses. © 2000 Elsevier Science Ltd. All rights reserved.

Keywords: Beam; Cantilever; Control; Piezoelectric; Vibration

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1. Introduction

Piezoelectric actuators have proved to be effective control devices for active control of structural vibrations and presently are in use in a wide range of engineering applications. Piezo materials can be integrated into various structural components and utilized for distributed sensing and active damping of vibrations. The reviews by Crawley (1994) and Rao and Sunar (1994) provide comprehensive overviews of the published work in this field and the books by Miu (1993) and Banks et al. (1996) treat the control aspects using piezoelectric materials.

One of the widely used piezo materials in active vibration control is piezoceramics because of their light weight, large useful bandwidth, mechanical simplicity, efficient conversion of electrical and mechanical energy, ability to be easily integrated with the structure and stiffness. The resulting intelligent structure can be controlled by implementing a suitable control algorithm with the piezoceramics providing the actuation. Feedback control algorithms such as rate feedback, independent modal space control and coupled mode control can be used in the vibration control of distributed structures (Hong et al., 1998). However, in the case of rate feedback control, there are some difficulties in deciding the control gain. Independent modal space control can be implemented using general control algorithms which require a fully distributed actuator and sensor. Therefore, it is a difficult technique to implement experimentally and the control system may change the structural characteristics (Hong et al., 1998). The coupled mode optimal control algorithm leads to a Ricatti equation, the solution of which is usually computationally time consuming. So far feedback control algorithms were used almost exclusively for the control of intelligent structures. Recent examples of this approach for vibration control can be found in Baz and Poh (1996), Chandrashekhara and Donthireddy (1997), Hong et al. (1998), Hsu et al. (1998), Kwak and Han (1998) and Ray (1998).

At this stage, there seems to be no studies using open-loop control approaches for vibration suppression in intelligent structures controlled by piezo actuators. The present study aims at filling the gap in this area using the optimal control theory of distributed parameter systems. The open-loop methods are known to have advantages of efficiency and accuracy as compared to closed-loop algorithms. Moreover, time delays in the implementation of a feedback control system may cause robustness problems (Datko, 1988) and in fact may destabilize the structure in certain cases (Sloss et al., 1992). A specific example of open-loop boundary control of a longitudinally vibrating rod was studied by Sadek et al. (1997) where the optimal control law was derived using a maximum principle. Boundary control of vibrating beams using a feedback control approach was given by Baz (1997) and Librescu and Na (1998a, 1998b, 1998c) where cantilever beams under velocity and acceleration feedback control were studied.

In the present study, the open-loop control results are obtained using a recently developed maximum principle for the optimal boundary control of one-dimensional structures (Sloss et al., 1998). For this purpose, the piezo-control problem is formulated as an optimal boundary control problem using the applied voltage as the control function. The objective function is specified as a quadratic functional of displacement, velocity and the expenditure of control force which is added as a penalty term. The open-loop control law is derived by introducing the adjoint problem and the related Hamiltonian in the form of a maximum principle. This approach leads to a system of coupled partial differential equations subject to initial, terminal and boundary conditions. Explicit solutions of the problem are obtained using eigenfunction expansions of the state and adjoint variables for a clamped-free beam with continuous piezoceramic actuators.

The effectiveness of the proposed control system is illustrated numerically by plotting the curves of deflection and velocity against time. These results indicate that the vibrations of the beam can be reduced substantially at a given terminal time by exercising open-loop piezo control. Furthermore, the reductions in displacement and velocity depend on the magnitude of the control moment which in turn depends on the penalty attached to the expenditure of control energy. Presently, an important issue involving piezo materials is their authority in exercising control on vibrating structures. In the present paper, we address this question by plotting non-dimensional graphs of control moment and piezo actuator thicknesses so that an assessment can be made of the required ratio of actuator to beam thickness to achieve a certain level of control for a given magnitude of applied voltage. The graphs also indicate the maximum control moment which can be achieved for a given set of input parameters.

2. Equation of motion for a piezoelectric beam

Consider a beam of length L, width b and height h_b sandwiched between two layers of piezoelectric material each of thickness h_p. Dimensions L, b and h_b are measured along the X-, Y-, and Z-axis, respectively, as shown in Fig. 1. The beam is cantilevered with $X = 0$ denoting the clamped end. When the control moment induced by the piezo actuators is not a function X, the differential equation governing the transverse vibrations of a piezoelectric beam is given by

$$\rho b W_{\bar{T}\bar{T}} + EIW_{XXXX} = 0, \quad 0 < X < L, 0 < \bar{T} < T_{\text{final}} \tag{1}$$

where ρ is the mass per unit length of the layered beam such that $\rho = \rho_b h_b + 2\rho_p h_p$, in which ρ_b and ρ_p denote the densities of the beam and piezo materials, respectively, EI is the bending stiffness of the beam including the piezoceramic layers, W is the transverse displacement of the beam and T_{final} is a given

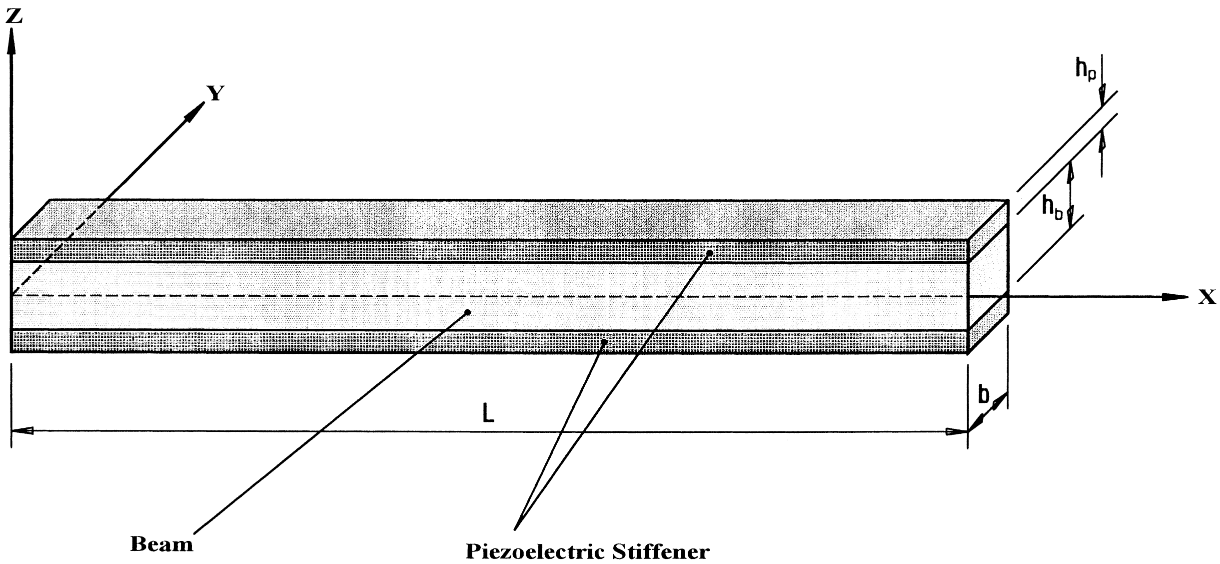

Fig. 1. Cantilever beam with distributed actuator layers.

terminal time. The subscripts $\bar{T}$ and X refer to differentiation with respect to these independent variables. The term EI is defined as the combined stiffness of the beam and stiffener–actuators and is given by:

$$EI = E_b I_b + \frac{2bE_p}{3}\left[\left(\frac{h_b}{2} + h_p\right)^3 - \left(\frac{h_b}{2}\right)^3\right] \tag{2}$$

in which $E_b I_b$ is the bending stiffness of the beam and E_p is the modulus of elasticity of the piezoelectric material. The beam is subject to the clamped-free boundary conditions

$$W(0, \bar{T}) = 0, \quad W_X(0, \bar{T}) = 0,$$

$$EIW_{XX}(L, \bar{T}) = r_1^a b d_{31} 2E_p V_p(\bar{T}), \quad W_{XXX}(L, \bar{T}) = 0, \qquad 0 \leq \bar{T} \leq T_{\mathrm{final}} \tag{3}$$

where $r_1^a = (h_b + h_p)/2$ is the effective moment arm, d_{31} is the piezo constant, and $V_p\bar{T}$ is the applied voltage. The initial conditions for this problem are taken as:

$$W(X, 0) = F(X), \quad W_{\bar{T}}(X, 0) = G(X), \qquad 0 < X < L \tag{4}$$

To derive a dimensionless equation of motion, the following expressions are introduced

$$w(x, t) = \frac{W(X, \bar{T})}{L}, \quad x = \frac{X}{L}, \quad t = \frac{\bar{T}}{L^2}\sqrt{\frac{EI}{\rho b}}, \quad T = \frac{T_{\mathrm{final}}}{L^2}\sqrt{\frac{EI}{\rho b}} \tag{5}$$

where $w(x, t)$, x, and t are the dimensionless transverse displacement, position, and time, respectively. Substituting Eq. (5) into Eqs. (1), (3) and (4) results in the following dimensionless equation of motion:

$$w_{tt} + w_{xxxx} = 0, \quad 0 < x < 1, 0 < t < T \tag{6}$$

subject to the boundary conditions

$$w(0, t) = 0, \quad w_x(0, t) = 0, \quad w_{xx}(1, t) = \alpha(t), \quad \text{and} \quad w_{xxx}(1, t) = 0, \qquad 0 \leq t \leq T \tag{7}$$

and initial conditions:

$$w(x, 0) = f(x), \quad w_t(x, 0) = g(x), \qquad 0 < x < 1 \tag{8}$$

where $f(x) = F(X)/L$ and $g(x) = L\sqrt{(\rho b)/(EI)}G(X)$ are non-dimensional functions. The moment control is then:

$$\alpha(t) = \frac{2r_1^a b d_{31} E_p L V_p(t)}{EI} \tag{9}$$

3. Optimal control problem

The objective of the control is to minimize the dynamic response of the beam with a limited expenditure of control energy. These objectives are expressed by defining a performance index given by

$$J(\alpha) = \int_0^1 \left[\mu_1 w^2(x, T) + \mu_2 w_t^2(x, T) \right] \mathrm{d}x + \int_0^T \mu_3 \alpha^2(t)\, \mathrm{d}t \tag{10}$$

where $\mu_1,\ \mu_2 > 0$, and $\mu_3 > 0$ are weight constants with $\mu_1 + \mu_2 > 0$. The last term in Eq. (10) is a penalty term on control energy.

The optimal control is defined as $\alpha^*(t) \in U_{\mathrm{ad}} = \{\alpha | \alpha \in L^2(0, T)\}$ such that

$$J(\alpha^*) = \min_{\alpha \in U_{\mathrm{ad}}} J(\alpha) \tag{11}$$

with $w(x, t)$ subject to Eqs. (6)–(9).

The solution of the control problem (11) is obtained by introducing an adjoint problem with the adjoint variable $v(x, t)$ satisfying the non-dimensional differential equation:

$$v_{tt} + v_{xxxx} = 0, \quad 0 < x < 1, 0 < t < T \tag{12}$$

with boundary conditions

$$v(0, t) = 0, \quad v_x(0, t) = 0, \quad v_{xx}(1, t) = 0, \quad v_{xxx}(1, t) = 0 \tag{13}$$

and terminal conditions

$$v(x, T) = -2\mu_2 w_t(x, T), \quad v_t(x, T) = 2\mu_1 w(x, T) \tag{14}$$

The maximum principle formulated in Sloss et al. (1998) states that $\alpha^*(t)$ is the optimal boundary control function of Eq. (11) if

$$\max_{\alpha \epsilon U_{\mathrm{ad}}} H\left[t; v_x^0(1, t), \alpha(t)\right] = H\left[t; v_x^0(1, t), \alpha^0(t)\right] \tag{15}$$

almost everywhere in $(0, 1) \times (0, T)$ where

$$H\left[t; v_x(1, t), \alpha(t)\right] = v_x(1, t)\alpha(t) - \mu_3 \alpha^2(t) \tag{16}$$

then

$$\alpha^*(t) = \alpha^0(t) \tag{17}$$

From Eqs. (16) and (17), it follows that the optimal control function $\alpha^*(t)$ is given by

$$\alpha^*(t) = \frac{1}{2\mu_3} v_x^0(1, t) \tag{18}$$

To determine the optimal control function $\alpha^*(t)$, $v_x^0(1, t)$ needs to be evaluated, such that $v^0(x, t)$ is the

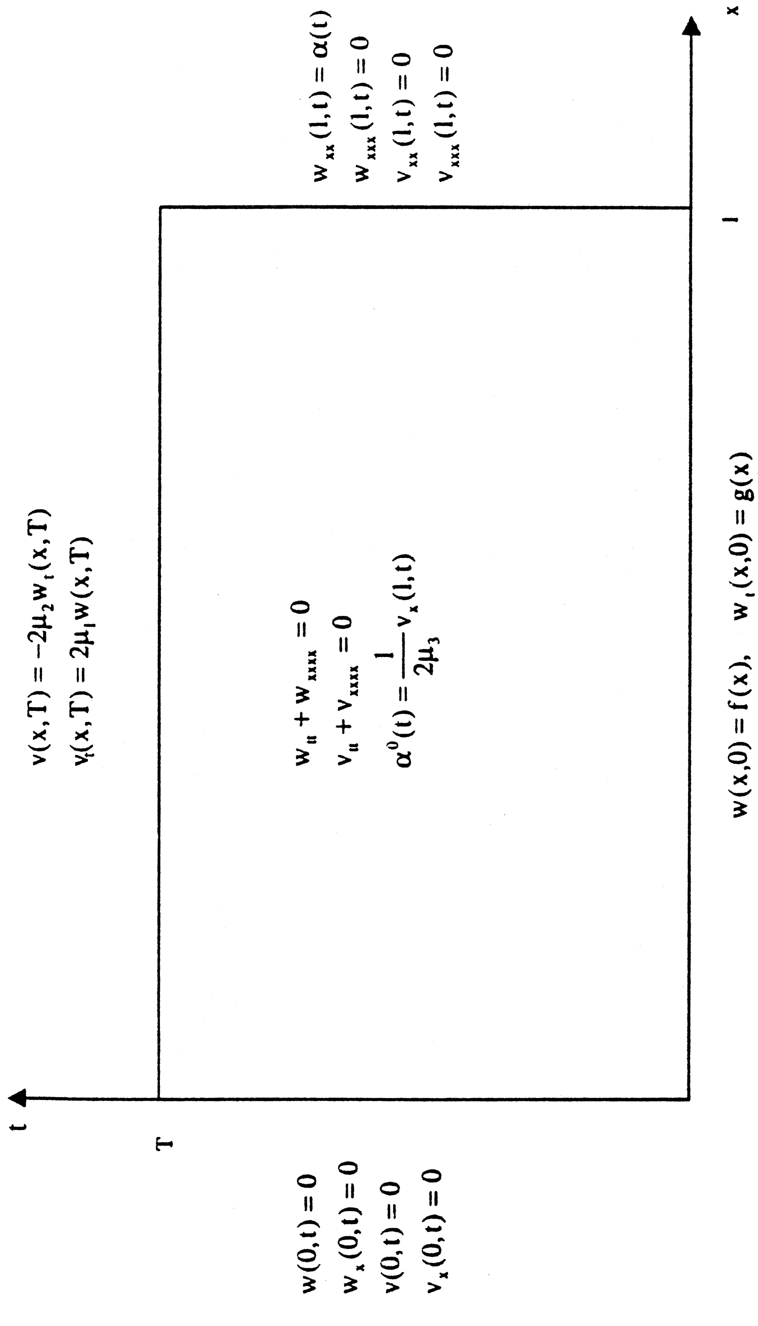

Fig. 2. Schematic presentation of the control problem on the space–time plane.

solution of the adjoint equation (12) subject to the boundary and terminal conditions (13) and (14). The solution of the adjoint problem, $v^0(x, t)$, and the solution of the piezoelectric beam, $w(x, t)$, are coupled through the terminal conditions and the optimality condition (18). Fig. 2 demonstrates a schematic presentation of the boundary control problem. The derivation of the adjoint operator, the terminal conditions, and the optimal boundary moment control are given in the next section.

4. Adjoint problem and the optimal control function

We note that

$$\int_0^T \int_0^1 (L\Delta w) v^0 \, \mathrm{d}x \, \mathrm{d}t = 0 \tag{19}$$

where L is an operator with $L(w) = w_{tt} + w_{xxxx}$, v^0 is the adjoint variable corresponding to the optimal boundary moment control α^0, and $\Delta w = w - w^0$, in which w^0 is the optimal displacement. Using the operator L gives $Lw = 0$ and $Lw^0 = 0$. Eq. (19) can be expanded as

$$\int_0^T \int_0^1 (\Delta w_{tt} + \Delta w_{xxxx}) v^0 \, \mathrm{d}x \, \mathrm{d}t = 0 \tag{20}$$

Integrating Eq. (20) twice by parts, one obtains

$$\begin{aligned} \int_0^T \int_0^1 (L\Delta w) v^0 \, \mathrm{d}x \, \mathrm{d}t = {} & \int_0^T \int_0^1 \left(\Delta w \, v^0_{tt} + \Delta w \, v^0_{xxxx}\right) \mathrm{d}x \, \mathrm{d}t + \int_0^1 \left[\Delta w_t v^0 - \Delta w \, v^0_t\right]_0^T \mathrm{d}x + \int_0^T \Big[\Delta w_{xxx} v^0 \\ & - \Delta w_{xx} v^0_x + \Delta w_x v^0_{xx} - \Delta w v^0_{xxx}\Big]_0^1 \mathrm{d}t \end{aligned} \tag{21}$$

where the first term on the right-hand side of Eq. (21) can be replaced by

$$\Delta w \, v^0_{tt} + \Delta w \, v^0_{xxxx} = \Delta w L^*(v^0) \tag{22}$$

in which $L^* = (\partial^2/\partial t^2) + (\partial^4/\partial x^4)$ is the *adjoint operator*.

Incorporating the initial conditions, the second term of Eq. (21) becomes

$$\int_0^1 [\underbrace{\Delta w_t(x, T) v^0(x, T)}_{(1)} - \underbrace{\Delta w(x, T) v^0_t(x, T)}_{(2)}] \, \mathrm{d}x = \mathrm{I} \tag{23}$$

after using the following relations:

$$\Delta w_t(x, 0) v^0(x, 0) = 0 \quad \text{since } w_t(x, 0) = g(x), \; w^0_t(x, 0) = g(x)$$

$$\Delta w(x, 0) v^0_t(x, 0) = 0 \quad \text{since} \quad w(x, 0) = f(x), \; w^0(x, 0) = f(x) \tag{24}$$

The last term in Eq. (21) expands to the following expression once the boundary conditions are incorporated

$$\int_0^T \big[\Delta w_{xxx}(1,t)v^0(1,t) - \Delta w_{xx}(1,t)v_x^0(1,t) + \Delta w_x(1,t)v_{xx}^0(1,t) - \Delta w(1,t)v_{xxx}^0(1,t)$$
$$- \Delta w_{xxx}(0,t)v^0(0,t) + \Delta w_{xx}(0,t)v_x^0(0,t) - \Delta w_x(0,t)v_{xx}^0(0,t) + \Delta w(0,t)v_{xxx}^0(0,t)\big]\,\mathrm{d}t$$
$$= \mathrm{II} \tag{25}$$

Eq. (25) can be simplified to obtain

$$\int_0^T \big[-\Delta\alpha(t)v_x^0(1,t)\big]\,\mathrm{d}t = \mathrm{II} \tag{26}$$

after the incorporation of the following boundary conditions

$$\Delta w(0,t) = 0, \quad \Delta w_x(0,t) = 0, \quad \Delta w_{xx}(1,t) = \Delta\alpha(t) = \alpha - \alpha^0, \quad \Delta w_{xxx}(1,t) = 0$$

$$v^0(0,t) = 0, \quad v_x^0(0,t) = 0, \quad v_{xx}^0(1,t) = 0, \quad v_{xxx}(1,t) = 0 \tag{27}$$

Moreover, rewriting Eq. (21) and using Eqs. (23) and (26) gives

$$\int_0^T\int_0^L \big[(L\Delta w)v^0 - \Delta w L^*(v^0)\big]\,\mathrm{d}x\,\mathrm{d}t = I - \int_0^T \Delta\alpha(t)v_x^0(1,t)\,\mathrm{d}t = 0 \tag{28}$$

With the use of Taylor series expansion and Eq. (10), the following can be defined

$$J(\alpha) - J(\alpha^0) = \int_0^1 \left[\overbrace{2\mu_1 w^0\Delta w}^{(2)} + \mu_1 r_1 + \overbrace{2\mu_2 w_t^0\Delta w_t}^{(1)} + \mu_2 r_2\right]\mathrm{d}x + \int_0^t \mu_3\left[\alpha^2(x,t) - \alpha^{0^2}(x,t)\right]\mathrm{d}t \tag{29}$$

where $r_1 > 0$ and $r_2 > 0$.

The terminal conditions are obtained by equating parts (1) and (2) of Eqs. (23) and (29). The *terminal conditions* then are

$$v^0(x,T) = -2\mu_2 w_t^0(x,T), \quad v_t^0(x,T) = 2\mu_1 w^0(x,T) \tag{30}$$

Thus, Eq. (29) can be written as

$$J(\alpha) - J(\alpha^0) = \int_0^1 (\mu_1 r_1 + \mu_2 r_2)\,\mathrm{d}x - \mathrm{I} + \int_0^T \mu_3\left[\alpha^2(t) - \alpha^{0^2}(t)\right]\mathrm{d}t \tag{31}$$

where from Eq. (28)

$$\mathrm{I} = \int_0^T \Delta\alpha(t)v_x^0(1,t)\,\mathrm{d}t \tag{32}$$

Deleting the first term on the right-hand side of Eq. (31), which is always positive, gives the inequality

$$J(\alpha) - J(\alpha^0) = \int_0^T \Big[-\Delta\alpha(t) v_x^0(1, t) + \mu_3 \big[\alpha^2(t) - \alpha^{0^2}(t)\big]\Big]\, \mathrm{d}t \geq 0 \tag{33}$$

which implies that

$$-\Delta\alpha(t) v_x^0(1, t) + \mu_3\big(\alpha^2 - \alpha^{0^2}\big) \geq 0 \tag{34}$$

This can be rewritten as

$$\alpha^0(t) v_x^0(1, t) - \mu_3 \alpha^{0^2} \geq \alpha(t) v_x^0(1, t) - \mu_3 \alpha^2 \tag{35}$$

A "Maximum Principle" is introduced to obtain the optimal moment function. Defining

$$H\big(t; v_x^0, \alpha\big) = \alpha v_x^0(1, t) - \mu_3 \alpha^2(t) \tag{36}$$

such that

$$\max_{\alpha \in U_{\mathrm{ad}}} H\big(t; v_x^0, \alpha\big) = H\big(t; v_x^0, \alpha^0\big) \tag{37}$$

Hence,

$$\max_{\alpha \in U_{\mathrm{ad}}} \big[\alpha v_x^0(1, t) - \mu_3 \alpha^2(t)\big] = \max_{\alpha \in U_{\mathrm{ad}}} \left[-\mu_3 \left\{ \frac{v_x^0(1, t)}{2\mu_3} - \alpha(t) \right\}^2 + \frac{1}{4\mu_3} v_x^{0^2}(1, t) \right] \tag{38}$$

if

$$\left[\frac{v_x^0(1, t)}{2\mu_3} - \alpha(t) \right]_{\alpha = \alpha^0} = 0 \tag{39}$$

and then

$$\alpha^0 = \frac{v_x^0(1, t)}{2\mu_3} \tag{40}$$

where α_0 is the optimal boundary moment control which maximizes the function in Eq. (36).

5. Method of solution

Consider the equation of motion of the cantilever beam in non-dimensional form

$$w_{tt} + w_{xxx} = 0, \quad 0 < x < 1, 0 < t < T \tag{41}$$

where the boundary and initial conditions are defined in Eqs. (7) and (8). The expression for the optimal control $\alpha_0(t)$ is given in terms of $v^0(1, t)$ in Eq. (40) where $v^0(x, t)$ is the solution of the adjoint problem:

$$v_{tt}^0 + v_{xxxx}^0 = 0 \tag{42}$$

subject to the boundary and terminal conditions

$$v^0(0, t) = 0,\ v_x^0(0, t) = 0,\ v_{xx}^0(1, t) = 0,\ v_{xxx}^0(1, t) = 0$$

$$v^0(x, T) = -2\mu_2 w_t^0(x, T),\ v_t^0(x, T) = 2\mu_1 w^0(x, T) \tag{43}$$

The above problem contains a non-homogeneous boundary condition in Eq. (7) and can be converted to a problem with homogeneous boundary conditions for ease of solution. The conversion is achieved by defining a new variable $\bar{w}$ such that

$$\bar{w}(x, t) = w(x, t) - s(x)\alpha(t) \tag{44}$$

where $s(x)$ is to be determined to obtain homogenous boundary conditions for $\bar{w}(x, t)$. The new problem is solved for $\bar{w}$ from the converted equation of motion

$$\bar{w}_{tt} + \bar{w}_{xxxx} = -s(x)\ddot{\alpha}(t) \quad 0 < x < 1, 0 < t < T \tag{45}$$

where the overdots refer to differentiation with respect to t. The boundary and initial conditions are expressed in terms of the newly defined variable, viz.,

$$\bar{w}(0, t) = 0, \quad \bar{w}_x(0, t) = 0, \quad \bar{w}_{xx}(1, t) = 0, \quad \bar{w}_{xxx}(1, t) = 0 \tag{46}$$

and

$$\bar{w}(x, 0) = f(x) - s(x)\alpha(0), \quad \bar{w}_t(x, 0) = g(x) - s(x)\dot{\alpha}(0) \tag{47}$$

where $f(x) = w(x, 0)$ and $g(x) = w_t(x, 0)$ are the previously defined initial conditions.

The solution of Eq. (45) is obtained in the form

$$\bar{w}(x, t) = \sum_{n=1}^{\infty} \bar{w}_n(t)\phi_n(x) \tag{48}$$

where $\phi_n(x)$ are the eigenfunctions of the uncontrolled cantilever beam problem which are expressed as

$$\phi_n(x) = -\left(\frac{\sinh \beta_n - \sin \beta_n}{\cosh \beta_n + \cos \beta_n}\right)\left(\sinh \beta_n x - \sin \beta_n x\right) + \left(\cosh \beta_n x - \cos \beta_n x\right) \tag{49}$$

in which β_n are the eigenvalues. The characteristic equation for the eigenvalues is $\cos \beta + \cosh \beta = -1$.

Similarly, the solution of the adjoint equation is expressed as

$$v^0(x, t) = \sum_{n=1}^{\infty} v_n(t)\phi_n(x) \tag{50}$$

where $v_n(t)$ is given by

$$v_n(t) = a_n \cos \lambda_n t + b_n \sin \lambda_n t \tag{51}$$

in which $\lambda_n = \beta_n^2$.

Expanding $s(x)$ in terms of the eigenfunctions $\phi_n(x)$ results in

$$s(x) = \sum_{n=1}^{\infty} s_n \phi_n(x) \tag{52}$$

where

$$s_n = \frac{\int_0^1 s(x)\phi_n(x)\,\mathrm{d}x}{\int_0^1 \phi_n^2(x)\,\mathrm{d}x} \tag{53}$$

Also,

$$\ddot{\alpha}^0(t) = \frac{1}{2\mu_3} v^0_{xtt}(1, t) = \frac{1}{2\mu_3} \sum_{m=1}^{\infty} \ddot{v}_m(t)\phi'_m(1) \tag{54}$$

where the prime denotes differentiation with respect to x. The function $\bar{w}_n(t)$ satisfies the differential equation

$$\ddot{\bar{w}}_n + \lambda_n^2 \bar{w}_n = -s_n \ddot{\alpha}(t) \tag{55}$$

and is expressed as

$$\bar{w}_n(t) = d_n \cos \lambda_n t + e_n \sin \lambda_n t - s_n \lambda_n^{-1} \int_0^t \sin \lambda_n (t - \xi)\ddot{\alpha}(\xi)\,\mathrm{d}\xi \tag{56}$$

The expansion of the initial conditions for $w(x, t)$ is given by:

$$\begin{Bmatrix} f(x) \\ g(x) \end{Bmatrix} = \sum_{n=1}^{\infty} \begin{Bmatrix} f_n \\ g_n \end{Bmatrix} \phi_n(x) \tag{57}$$

where

$$\begin{Bmatrix} f_n \\ g_n \end{Bmatrix} = \frac{\int_0^1 \{f(x)/g(x)\}\phi_n(x)\,\mathrm{d}x}{\int_0^1 \phi_n^2(x)\,\mathrm{d}x} \tag{58}$$

Hence, using the relation $v_m(0) = a_m$, the first initial condition in Eq. (47) gives

$$\bar{w}_n(0) = d_n = f_n - s_n\alpha(0) = f_n - (s_n/2\mu_3) \sum_{m=1}^{\infty} a_m \phi'_m(1) \tag{59}$$

Likewise, since $\dot{v}_m(0) = b_m\lambda_m$, the second initial condition in Eq. (47) gives

$$\dot{\bar{w}}_n(0) = e_n\lambda_n = g_n - s_n\dot{\alpha}(0) = g_n - \left(s_n/2\mu_3\right)\sum_{m=1}^{\infty} b_m\lambda_m\phi'_m(1) \tag{60}$$

Eq. (55) and its time derivative can be written as

$$\bar{w}_n^0(t) = \sum_{m=1}^{\infty}\left[M_{nm}(t)a_m + N_{nm}(t)b_m\right] + f_n\cos\lambda_n t + (g_n/\lambda_n)\sin\lambda_n t \tag{61}$$

$$\dot{\bar{w}}_n^0(t) = \sum_{m=1}^{\infty}\left[\dot{M}_{nm}(t)a_m + \dot{N}_{nm}(t)b_m\right] - f_n\lambda_n\sin\lambda_n t + g_n\cos\lambda_n t \tag{62}$$

where the variables M_{nm} and N_{nm} are given by

$$M_{nm}(t) = \left[s_n\phi'_m(1)/2\mu_3\right]\left[-\cos\lambda_n t + \left(\lambda_m^2/\lambda_n\right)\int_0^t \sin\lambda_n(t-\xi)\cos\lambda_m\xi\,\mathrm{d}\xi\right] \tag{63}$$

$$N_{nm}(t) = \left[s_n\phi'_m(1)/2\mu_3\right]\left[-(\lambda_m/\lambda_n)\sin\lambda_n t + \left(\lambda_m^2/\lambda_n\right)\int_0^t \sin\lambda_n(t-\xi)\sin\lambda_m\xi\,\mathrm{d}\xi\right] \tag{64}$$

The terminal conditions in terms of $\bar{w}$ at $t = T$ become

$$v^0(x, T) = -2\mu_2\left[\bar{w}_t^0(x, T) + s(x)\dot{\alpha}^0(T)\right] \tag{65}$$

$$v_t^0(x, T) = 2\mu_1\left[\bar{w}^0(x, T) + s(x)\alpha^0(T)\right] \tag{66}$$

$$v_n^0(T) = -2\mu_2\left[\dot{\bar{w}}_n^0(T) + s_n\left(\frac{1}{2\mu_3}\right)\sum_{m=1}^{\infty}\dot{v}_m^0(T)\phi'_m(1)\right] \tag{67}$$

$$\dot{v}_n^0(T) = 2\mu_1\left[\bar{w}_n^0(T) + s_n\left(\frac{1}{2\mu_3}\right)\sum_{m=1}^{\infty}v_m^0(T)\phi'_m(1)\right] \tag{68}$$

A system of linear equations can be formed to solve for the constants a_n and b_n from the terminal conditions. The system of the linear equations is in the form:

$$\sum_{m=1}^{\infty}\Gamma_{nm}^1 a_m + \sum_{m=1}^{\infty}\Gamma_{nm}^2 b_m = \gamma_{1n}, \quad n = 1, 2, 3, \ldots \tag{69}$$

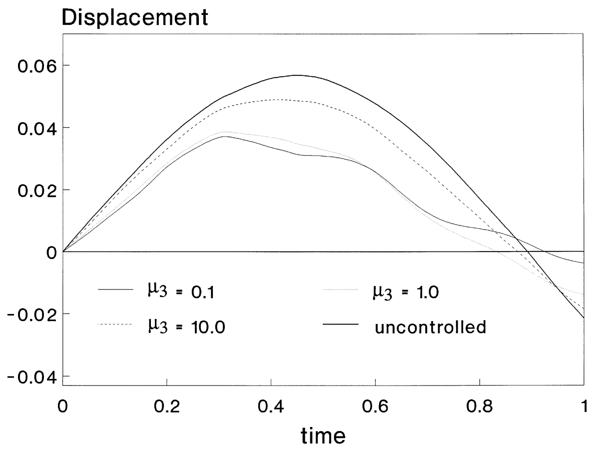

Fig. 3. Controlled and uncontrolled displacements versus time at $x = 1.0$ with $T = 1.0$.

$$\sum_{m=1}^{\infty} \Gamma_{nm}^3 a_m + \sum_{m=1}^{\infty} \Gamma_{nm}^4 b_m = \gamma_{2n}, \quad n = 1, 2, 3, \ldots \tag{70}$$

where the Γ_{nm} constants are given by

$$\Gamma_{nm}^1 = \delta_{nm} \cos \lambda_n T + 2\mu_2 \dot{M}_{nm}(T) - (s_n \mu_2 / \mu_3) \lambda_m \sin \lambda_m T \phi_m'(1) \tag{71}$$

$$\Gamma_{nm}^2 = \delta_{nm} \sin \lambda_n T + 2\mu_2 \dot{N}_{nm}(T) + (s_n \mu_2 / \mu_3) \lambda_m \cos \lambda_m T \phi_m'(1) \tag{72}$$

$$\Gamma_{nm}^3 = -\delta_{nm} \lambda_n \sin \lambda_n T - 2\mu_1 M_{nm}(T) - (s_n \mu_1 / \mu_3) \cos \lambda_m T \phi_m'(1) \tag{73}$$

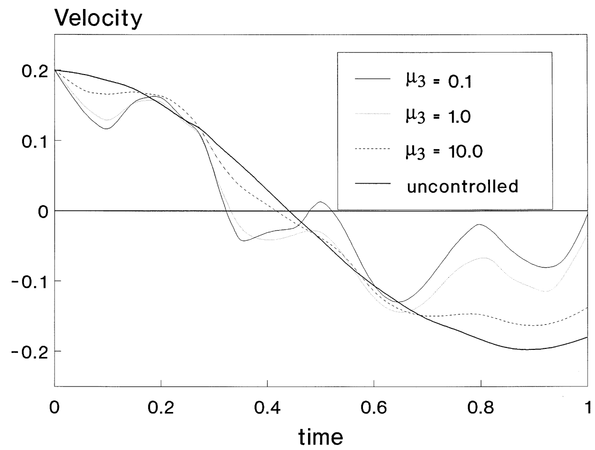

Fig. 4. Controlled and uncontrolled velocities versus time at $x = 1.0$ with $T = 1.0$.

$$\Gamma^4_{nm} = \delta_{nm}\lambda_n\cos\lambda_n T - 2\mu_1 N_{nm}(T) - (s_n\mu_1/\mu_3)\lambda_m\sin\lambda_m T\phi'_m(1) \tag{74}$$

in which $\delta_{nm} = 1$ if $n = m$ and $\delta_{nm} = 0$ otherwise, and the γ's are given by

$$\gamma_{1n} = -2\mu_2\left[-f_n\lambda_n\sin\lambda_n T + g_n\cos\lambda_n T\right] \tag{75}$$

$$\gamma_{2n} = 2\mu_1\left[f_n\cos\lambda_n T + (g_n/\lambda_n)\sin\lambda_n T\right] \tag{76}$$

Obtaining the displacement and velocity involves finding the constants a_n and b_n and then determining d_n and e_n from Eqs. (59) and (60). Once all constants are determined, $\bar{w}^0_n$ and $\dot{\bar{w}}^0_n$ are obtained from Eqs. (61) and (62).

6. Numerical results

The numerical results are given for the cantilever beam subject to the initial impact conditions $w(x, 0) = 0$ and $w_t(x, 0) = 0.1\phi_1(x)$ where $\phi_1(x)$ corresponds to the first eigenfunction of the beam given by Eq. (49). All the results are given for a terminal time $T = 1.0$ with the weighting coefficients μ_1 and μ_2 in the performance index (10) specified as $\mu_1 = \mu_2 = 1.0$. The function $s(x)$ used to transform the problem with non-homogeneous boundary conditions to the one with homogeneous ones was chosen as $s(x) = x^2/2$. Results for the displacement and velocity are given at the tip of the cantilever, i.e., $x = 1.0$.

Fig. 3 shows the curves of displacement plotted against time for the uncontrolled beam and the controlled beams with $\mu_3 = 0.1$, 1.0 and 10.0. At the terminal time $T = 1.0$, the deflection of the uncontrolled beam is $|w_u(1, 1)| = 0.215$ and the corresponding value for the controlled beam with $\mu_3 = 0.1$ is $|w_c(1, 1)| = 0.038$ indicating a substantial reduction of the deflection at the specified terminal time.

Corresponding results for velocity are given in Fig. 4. At $T = 1.0$, the magnitude of the velocity depends on the penalty applied on the control functional as indicated by μ_3 with $\mu_3 = 0.1$ (the smallest

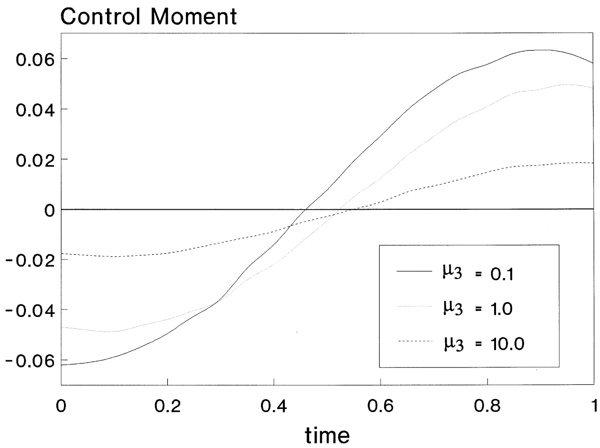

Fig. 5. Control moment $\alpha(t)$ versus time with $T = 1.0$.

penalty) giving the lowest velocity. Again the decrease in the controlled velocities at $T = 1.0$ is substantial as compared to the uncontrolled one.

The non-dimensional optimal control function given by $\alpha(t)$ is shown in Fig. 5 as a function of time for $\mu_3 = 0.1$, 1.0 and 10.0. As expected, the lowest μ_3 leads to the highest control moment. The maximum $\alpha(t)$ is 0.0623 for $\mu_3 = 0.1$.

The above results were given for non-dimensional quantities and the question remains whether a piezo actuator will be able to deliver enough control moment to damp out the vibrations at the given terminal time. More specifically, it is observed that to implement the control law (18) with $\mu_3 = 0.1$, the piezo actuator should be capable of achieving an $\alpha(t)$ value of 0.0623 as indicated in Fig. 5.

To answer this question, the relations between the maximum α, applied voltage and the thicknesses of the piezoceramic layers and the beam have to be investigated. Towards this end, we define the following non-dimensional quantities

$$H_b = h_b/L, \quad H_p = h_p/L, \quad h_r = h_p/h_b \tag{77}$$

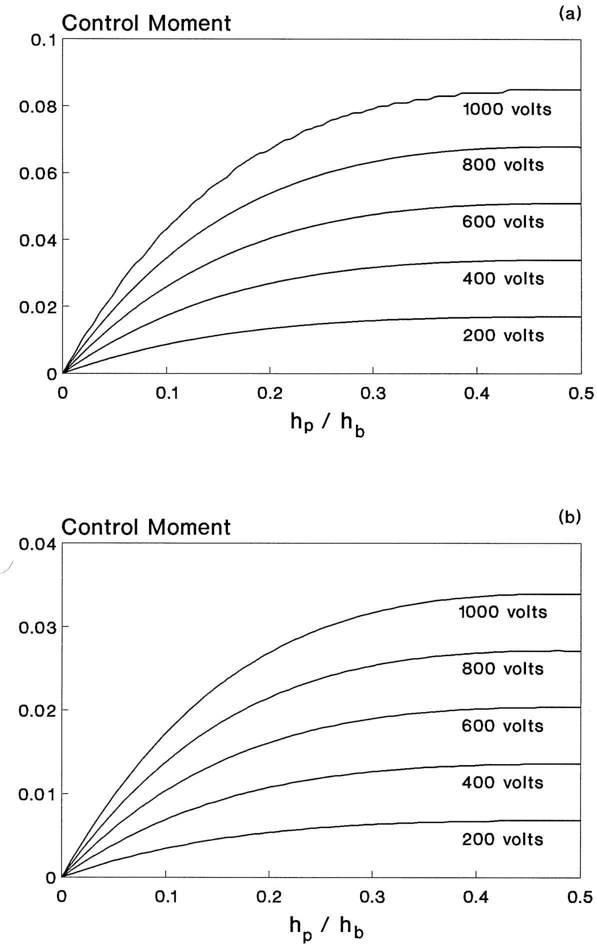

Fig. 6. Maximum control moment versus h_p/h_b for (a) $h_b/L = 0.002$ and (b) $h_b/L = 0.005$.

By substituting EI from Eq. (2) into $\alpha(t)$ in Eq. (9), we obtain

$$\alpha(t) = Cd_{31}\ V_{\mathrm{p}}(t)/h_{\mathrm{b}} \tag{78}$$

where C is the non-dimensional constant given by

$$C = \left(h_{\mathrm{r}} + h_{\mathrm{r}}^2\right)E_{\mathrm{p}}/\left[H_{\mathrm{b}}\left(E_{\mathrm{b}}/12 + (2/3)E_{\mathrm{p}}\left[(1/2 + h_{\mathrm{r}})^3 - (1/8)\right]\right)\right] \tag{79}$$

The maximum voltage that can be applied to a piezoceramic material is specified as 500–1000 V/mm of piezo layer thickness (Morgan Matroc, Inc., 1993). Let this maximum value be denoted by V. Then

$$\max_{0\le t\le T} V_{\mathrm{p}}(t) = Vh_{\mathrm{p}} \times 10^3 \tag{80}$$

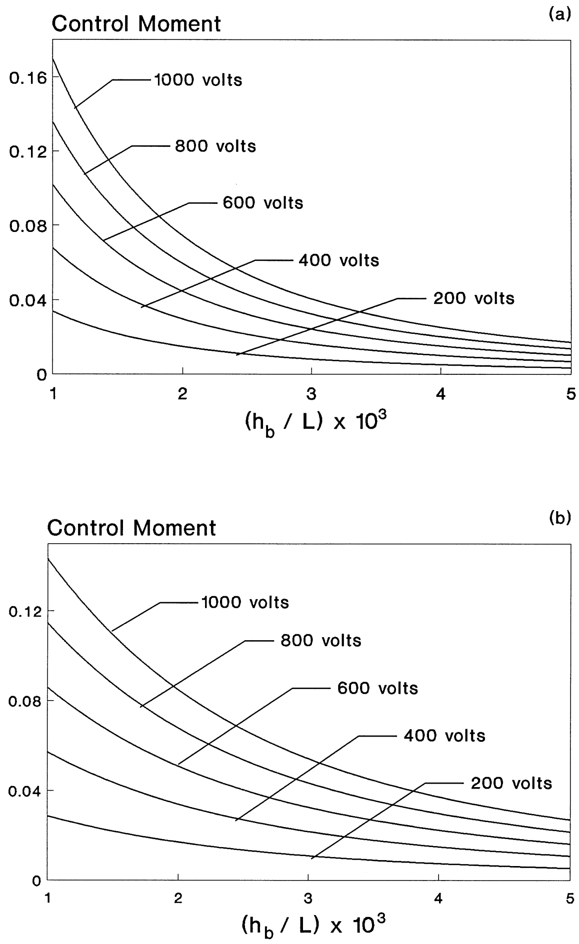

Fig. 7. Maximum control moment versus h_{b}/L for (a) $h_{\mathrm{p}}/L = 0.0005$ and (b) $h_{\mathrm{p}}/L = 0.0010$.

where h_p is given in meters and converted to millimeters by multiplying it by 10^3. Thus, from Eqs. (78) and (80), it follows that

$$\alpha_{max} = Ch_r d_{31} V \times 10^3 \tag{81}$$

where

$$\alpha_{max} = \max_{0 \le t \le T} \alpha(t)$$

Next, the behavior of α_{max} (control moment) is investigated with respect to $h_r = h_p/h_b$ and $H_b = h_b/L$. For this purpose, let the beam material be specified as titanium with $E_b = 114$ GPa and the piezoceramic as G-1195 with $E_p = 63$ GPa and $d_{31} = -166 \times 10^{-12}$ m/V.

Fig. 6 shows the curves of α_{max} vs. $h_r = h_p/h_b$ for $h_b/L = 0.002$ and 0.005. It is observed that as h_r increases, α_{max} increases, but this increase tapers off after about $h_r = 0.2$. A comparison of Fig. 6a and b indicates that increasing the beam thickness, i.e., $H_b = h_b/L$ reduces α_{max}. In the control problem studied above, a value of $\alpha_{max} = 0.0623$ was needed in the control process for $\mu_3 = 0.1$. This can be realized by choosing $h_b/L = 0.002$, and then finding the minimum applied voltage and h_r required to obtain α_{max} from Fig. 6a. If the beam thickness is $h_b/L = 0.005$ (Fig. 6b), the required maximum moment cannot be realized using the above piezoceramic material.

The dependence of α_{max} on h_b/L is investigated in Fig. 7 for $H_p = h_p/L = 0.0005$ and 0.0010. As the beam thickness increases, the maximum control moment drops with this drop tapering off after about $h_b/L = 0.25$. As h_b/L increases, the differences in the α_{max} values as a function of V become quite small. One can use Fig. 7a and b to determine the minimum voltage required to achieve a certain α_{max} for given values of piezo and beam thicknesses.

A comparison of Fig. 7a and b indicates that increasing h_p/L ratio decreases α_{max} value for beams of $h_b/L < 0.0015$, but the effect is opposite for thicker beams of $h_b/L > 0.0015$.

Figs. 6 and 7 give an indication of the beam and piezo dimensions as well as the voltage required to achieve a certain maximum moment specifically for titanium and G-1195 piezoceramic combination. Similar figures can be plotted to determine the required voltage and piezo thicknesses to deliver a certain control force or moment for given active and passive material combinations.

7. Conclusions

Vibrations of a beam were damped out using optimally controlled piezo actuation and the amount of control moment that can be obtained using piezoceramic materials was determined in terms of beam dimensions and applied voltage for a titanium beam and G-1195 piezoceramic actuator. An open-loop control approach was adopted to derive the control law as opposed to using a feedback control approach. The problem formulation leads to a boundary control problem as the piezo actuation term appears in the boundary conditions as a moment function. Taking the performance index as a quadratic functional of displacement and velocity with the expenditure of control energy attached as a penalty term, the control law was derived in terms of an adjoint variable using a maximum principle developed by Sloss et al. (1998). Analytical solutions of the problem were given for a cantilever beam. If a forcing function were present in the state equation, Eq. (1), the coefficients in the optimal control voltage would change due to the fact that these are calculated from the terminal conditions which are in terms of the state variable and its derivative at the terminal time.

It was observed that the proposed control mechanism based on piezo actuation and open-loop control mechanism was effective in reducing the displacement and vibration of the beam and this reduction depended on the amount of control energy spent during the control process.

A related problem when using piezoelectric materials for vibration damping is whether piezo actuation will be sufficiently strong to deliver the amount of force required in the control process. In other words, the piezo actuators should be capable of generating the maximum force or moment that is needed for controlling the vibrations. This issue was dealt with by studying the relations between four quantities, namely, maximum actuator moment, piezo layer thickness, beam thickness and applied voltage. The numerical results given for this purpose enable one to determine the required dimensions of the piezo material and the beam, and the voltage once the maximum control moment is specified.

Acknowledgements

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